Cash Flow and Unpaid Claim Runoff Estimates Using Mack and Merz-Wüthrich Models

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Abstract

Motivation. For both Solvency II and IFRS 17 the actuary can use unpaid claim variability estimates for cash flows and the runoff of unpaid claims in addition to the more widely used accident year view of the unpaid claims. Under Solvency II, the concept of the one-year time horizon adds a new dimension to the estimates for unpaid claim distributions. While other models are also available, this paper will focus on the closed form model developed by Mack, modified by Merz & Wüthrich to address the 1-year time horizon, and on its extension for 2-year, 3-year, etc. time windows that can be reconciled back to the original Mack formulas.

Method. This paper is based on a review of the foundational Mack and Merz-Wüthrich formulas and their decomposition into process variance and parameter uncertainty, per future diagonal. The decompositions are then used to show how modifications to the accident year formulas can be used to calculate the standard deviations for cash flow and unpaid claim runoff estimates. In addition, an alternative view of the covariance adjustment is developed to aide comparisons with other models.

Results. England, Verrall & Wüthrich propose using the N-year extrapolation of the Merz-Wüthrich formulas for risk margin estimates using the cost of capital method. In this paper, we will discuss how the original formulas can be modified to better fit the Solvency II time horizon concept.

Conclusions. The paper will demonstrate that while the Merz-Wüthrich formulas (and by extension the England, Verrall & Wüthrich formulas) are an elegant bridge between the 1-year time horizon and the ultimate time horizon used by Mack, there is an alternative formulation for the runoff of the N-year time horizon that better fits the Solvency II environment. The paper also includes cash flow and unpaid claim runoff formulations for Mack that can be used for IFRS 17.

Availability. In lieu of technical appendices, companion Excel workbooks are included that illustrate the calculations described in this paper. The companion materials are summarized in the Supplementary Materials section and are available by emailing the author.

Keywords. Reserve variability, chain ladder, prediction error, mean square error of prediction, cost of capital, risk margin, risk adjustment, Solvency II, IRFS 17, value at risk, tail value at risk, one-year time horizon.

1. INTRODUCTION

While there is now a large and growing volume of models that can be used for reserve variability estimates, one of the foundational models was introduced by Mack [5] in 1993. Because it is a closed form solution, which can be easily adapted in an Excel function or reserving software, it has gained widespread use.

The Solvency II regulatory regime in Europe introduced the concept of the 1-year time horizon and Merz & Wüthrich [7] took up the challenge of modifying the Mack formulas to directly estimate the reserve variability of the claim development result for a 1-year time horizon. Similar to the Mack formulas, the Merz-Wüthrich models have gained widespread use for Solvency II.

For both the Mack and Merz-Wüthrich formulas, the papers only focus on an accident year view of the claim development, which is natural as this is the primary configuration for reserving data. Fortunately, as all the "parts" are included in the formulas it is a natural extension of these models to work out the calendar year formulas for calculating the variance of the cash flows and unpaid claim runoff. In addition to typical uses, examining both of these in more detail helps to decompose the 1-year time horizon, which includes both parameter and process variance for the next calendar year, to estimate possible outcomes, and only parameter variance for the remaining future calendar years, to estimate reserves contingent on the possible outcomes in the next calendar year (i.e., over a 1-year time horizon).

1.1 Research Context

The model developed by Mack is widely used and understood by actuaries around the world and the Mack papers are well supported with derivations and proofs. Thus, this paper will focus on a high-level discussion of the modeling framework and will not reproduce the derivations and proofs as the reader can find these in the original papers.

The model developed by Merz-Wüthrich is widely used and understood in Europe, but less well known outside of Europe. The Merz & Wüthrich [7, 8] papers are similarly well supported with derivations and proofs that will not be reproduced in this paper.

This family of models is a distribution free method for calculating the variance of the chain ladder (CL) method by combining the process variance and parameter variance components of the mean squared error of prediction (MSEP):

$$\text{MSEP} \approx \sqrt{\textit{process variance} + \textit{parameter variance}} \tag{1.1}$$

The use of colors for the process variance and parameter variance components of the formulas is useful for clarifying the calculations and tracing the components through the various formulas.

1.2 Objective

The calendar year view of the standard formulas is an important addition to the actuarial literature to support cash flow and unpaid claim runoff calculations for different regulatory and financial reporting regimes, such as Solvency II and IRFS 17, in addition to enterprise risk management uses. A recent paper by England, Verrall & Wüthrich [3] examines how the ultimate and time horizon views of the Mack and Merz-Wüthrich models, respectively, are connected.

In England, Verrall & Wüthrich [3], the authors propose that the runoff of the time horizons using the Merz-Wüthrich formulas is ideal for uses such as the runoff of the capital for the cost of capital method of calculating a risk margin. We will examine this proposed use of the Merz-Wüthrich model and propose an alternative approach.

1.3 Outline

The remainder of the paper proceeds as follows. Section 2 will provide an overview of the notation used in the paper. In Section 3, the Mack model is described and then additional formulas for calculating the variance of the cash flows and runoff of the unpaid claims are specified. Next, Section 4 will focus on the Merz-Wüthrich models, which include the runoff of the time horizon beyond year one. Similar to the Mack model discussion, the cash flow formulas will be specified. Then, in Section 5 alternative formulas for the runoff of the time horizon beyond year one will be proposed. Finally, Section 6 will discuss conclusions based on results applying the formulas to a real dataset.

2. NOTATION

The notation in this paper is from the CAS Working Party on Quantifying Variability in Reserve Estimates Summary Report [1] since it is intended to serve as a basis for further research. Many models visualize loss data as a two-dimensional array, (w, d), with accident period or policy period w and development age d (think w = "when" and d = "delay").[1] For this discussion, it is assumed that the loss information available is an "upper triangular" subset for rows $w = 1,2, \ldots, n$ and for development ages $d = 1,2, \ldots, n$. The "diagonal" for which $w + d -$

[1] For a more complete explanation of this two-dimensional view of the loss information, see the *Foundations of Casualty Actuarial Science* [4], Chapter 5, particularly pages 210-226.

1 equals the constant, k, represents the loss information for each accident period w as of accounting period k.[2]

For purposes of including tail factors, the development beyond the observed data for periods $d = n+1, n+2, \ldots, u$, where u is the ultimate time period for which any claim activity occurs – i.e., u is the period in which all claims are final and paid in full – must also be considered.

The paper uses the following notation for certain important loss statistics:

$c(w,d)$:	cumulative loss from accident year w as of age d.[3]
$q(w,d)$:	incremental loss for accident year w from $d-1$ to d.
$c(w,n) = U(w)$:	total loss from accident year w when claims are at ultimate values at time n, or with tail factors.[4]
$c(w,u) = U(w)$:	total loss from accident year w when claims are at ultimate values at time u.
$R(w)$:	future development after age $n-w+1$ for accident year w, i.e., $= U(w) - c(w, n-w+1)$.
$F(d)$:	factor applied to $c(w,d)$ to estimate $c(w,d+1)$.
$e(w,d)$:	a random fluctuation, or error, which occurs at the w, d cell.
$E(x)$:	the expectation of the random variable x.
$Var(x)$:	the variance of the random variable x. Or, alternatively σ_x^2.
σ_x:	the standard deviation of the random variable x.
$\hat{x}$:	an estimate of the parameter x.
N:	the total number of accident years.[5]

[2] Some authors define $d = 0,1,\ldots,n-1$ which intuitively allows $k = w$ along the diagonals, but in this case the triangle size is $n \times n-1$ which is not intuitive. With $d = 1,2,\ldots,n$ as defined in this paper, the triangle size $n \times n$ is intuitive, while $k = w + d - 1$ along the diagonals is less intuitive but still works. A way to think about this which helps tie everything together is to assume the w variables are the beginning of the accident periods and the d variables are at the end of the development periods. Thus, if years are used then cell $c(n,1)$ represents accident year n evaluated at 12/31/ n, or essentially 1/1/ $n+1$.

[3] The use of accident year is for ease of discussion. All of the discussion and formulas that follow could also apply to underwriting year, policy year, report year, etc. Similarly, year could also be half-year, quarter or month.

[4] This would imply that claims reach their ultimate value without any tail factor. This is generalized by changing n to $u = n + t$, where t is the number of periods in the tail.

[5] In a typical triangle the number of accident years, N, is the same as the number of development periods, n, but the number of development periods can be longer and even when they are the same using N vs. n helps visualize the calculations in the formulas.

The notation does not distinguish paid vs. incurred, but if this is necessary, capitalized subscripts P and I could be used. The cumulative known data, D, used in the formulas in this paper can be illustrated as follows:

		d					
		1	2	3	…	n-1	n
w	1	c(1,1)	c(1,2)	c(1,3)	…	c(1,n-1)	c(1,n)
	2	c(2,1)	c(2,2)	c(2,3)	…	c(2,n-1)	
	3	c(3,1)	c(3,2)	c(3,3)			
	…	…	…				
	N-1	c(N-1,1)	c(N-1,2)				
	N	c(N,1)					

To better illustrate the perspectives related to time between the Mack and Merz-Wüthrich models, the following notation and terms are used:

t: "at time" is equivalent to the valuation date used for financial accounting.

T: "time horizon" is the period for which the full distribution, including both process and parameter variance, is estimated.

T': "time window" is the period between the valuation date and the time when only the parameter variance is estimated.

3. MACK MODEL

Mack uses the common CL loss development model and demonstrates that, under specific assumptions, the best estimate of the age-to-age factors is the all-year volume weighted average:

$$\hat{F}(d) = \frac{\sum_{j=1}^{N-d} c(j, d+1)}{\sum_{j=1}^{N-d} c(j, d)} \tag{3.1}$$

Further, given the best estimate of the age-to-age factors, the best estimate of the ultimate value, given the known data, is calculated from the product of the age-to-age factors.

$$E[\hat{c}(w,n)|D] = c(w,d) \times \hat{F}(d) \times \hat{F}(d+1) \times \cdots \times \hat{F}(n-1) \tag{3.2}$$

3.1 Model Assumptions

For Mack's distribution free estimates of the variance, the formulas rest on three key assumptions. The first assumption is that the expected value of the next future cumulative value is the product of the previous cumulative value and the age-to-age factor:

$$E[\hat{c}(w, d+1)|D] = c(w, d) \times \hat{F}(d) \tag{3.3}$$

The second assumption is that the accident years are independent of one another:

$$\{c(i,1), c(i,2), \ldots, c(i,n)\} \ \& \ \{c(j,1), c(j,2), \ldots, c(j,n)\} \text{ are independent for all } i \neq j \tag{3.4}$$

The third assumption is that the variance of the next cumulative is proportional to the cumulative value:

$$Var[\hat{c}(w, d+1)|D] = c(w, d) \times \sigma_d^2 \tag{3.5}$$

Testing of these assumptions has been discussed by Mack and other authors so, similar to the proofs, the details of this testing are not included with this paper.[6]

3.2 Uncertainty by Accident Year

Building on these assumptions, the first step in calculating the total variance by accident year is to calculate the variance of the development periods, σ_d^2. Mack demonstrates that the unbiased estimator of the variance of the development periods is calculated using formula (3.6).

$$\hat{\sigma}_d^2 = \frac{1}{N-d-1} \times \sum_{j=1}^{N-d} c(j,d) \times \left\{\frac{c(j,d+1)}{c(j,d)} - \hat{F}(d)\right\}^2 ; \ 1 \leq d \leq n-2 \tag{3.6}$$

The interpretation of formula (3.6) is straightforward as this is the commonly used weighted standard deviation of the age-to-age factors, noting that $N-d$ is the number of individual age-to-age factors for development period d. For the last age-to-age factor, if $\hat{F}(n-1) = 1$ then we could assume that the development is finished and set $\hat{\sigma}_{n-1}^2 = 0$. However, if $\hat{F}(n-1) \neq 1$ then Mack suggested that the value for $\hat{\sigma}_{n-1}^2$ could be calculated by extrapolating using a loglinear regression of $\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_{n-2}$. Mack also suggested a simpler approach using formula (3.7), which is used in the examples that follow.

$$\hat{\sigma}_{n-1}^2 = \min[\frac{\hat{\sigma}_{n-2}^4}{\hat{\sigma}_{n-3}^2}, \min\{\hat{\sigma}_{n-3}^2, \hat{\sigma}_{n-2}^2\}] \tag{3.7}$$

Using the estimated variances by development period, Mack then demonstrates that the MSEP for the reserves by accident year can be calculated using formula (3.8).

$$Var\left[\hat{R}(w)\right] = \hat{c}(w,n)^2 \times \sum_{d=n+1-w}^{n-1} \frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \left\{\frac{1}{\hat{c}(w,d)} + \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right\} \tag{3.8}$$

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[6] For example, see Venter [10].

Reviewing the formula for the variance of the unpaid claims by accident year, (3.8), we can distinguish between the process variance component, which is the variance of the column of observed development factors, and the parameter variance component, which is the variance of the calculated weighted average development factors.

3.3 Total Uncertainty

To calculate the total variance for all accident years combined, we can rely on basic principles of statistics as the unpaid claim estimates are assumed to be the expected values, so the total estimated unpaid claims is the sum of the estimated unpaid claims by accident year, as shown in formula (3.9).

$$\hat{R}(tot) = \hat{R}(2) + \hat{R}(3) + \cdots + \hat{R}(N) \tag{3.9}$$

Similarly, the total variance for all accident years is the sum of the variances plus 2 times the covariance, as shown in formula (3.10).

$$Var\left[\hat{R}(tot)\right] = Var\left[\hat{R}(2)\right] + Var\left[\hat{R}(3)\right] + \cdots + Var\left[\hat{R}(N)\right] + 2 \times CoVariance \tag{3.10}$$

Using these basic principles of statistics, Mack developed the formula for the total variance, as shown in formula (3.11), which completes the modeling framework.[7]

$$Var\left[\hat{R}(tot)\right] = \sum_{w=2}^{N} \left\{ Var\left[\hat{R}(w)\right] + 2\hat{c}(w,n)\left(\sum_{i=w+1}^{N} c(i,n)\right) \sum_{d=n+1-w}^{n-1} \left(\frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right)\right\} \tag{3.11}$$

It is convenient to segregate the "bottom" or covariance portion of (3.11) when showing the results of the Mack calculations as this makes it easier for the user to quickly calculate the total uncertainty with and without the covariance adjustment (CVA) – i.e., assuming no correlation in the accident years. To illustrate the calculations in all formulas in this paper, we will use the Taylor & Ashe [9] data as our sample data, shown in Table 3.1, since it is used in many other papers.

[7] In some sense this modeling framework is not yet complete, as it does not include the tail variability. For ease of exposition, the tail variability is ignored in the paper but for completeness the companion Excel files include tail variability. The companion Excel files also allow the user to include exposure adjustments and exclude outliers.

Table 3.1 – Sample Data Triangle

	d										
		1	2	3	4	5	6	7	8	9	10
w	1	357,848	1,124,788	1,735,330	2,218,270	2,745,596	3,319,994	3,466,336	3,606,286	3,833,515	3,901,463
	2	352,118	1,236,139	2,170,033	3,353,322	3,799,067	4,120,063	4,647,867	4,914,039	5,339,085	
	3	290,507	1,292,306	2,218,525	3,235,179	3,985,995	4,132,918	4,628,910	4,909,315		
	4	310,608	1,418,858	2,195,047	3,757,447	4,029,929	4,381,982	4,588,268			
	5	443,160	1,136,350	2,128,333	2,897,821	3,402,672	3,873,311				
	6	396,132	1,333,217	2,180,715	2,985,752	3,691,712					
	7	440,832	1,288,463	2,419,861	3,483,130						
	8	359,480	1,421,128	2,864,498							
	9	376,686	1,363,294								
	10	344,014									
$\hat{F}(d)$		3.4906	1.7473	1.4574	1.1739	1.1038	1.0863	1.0539	1.0766	1.0177	
$\hat{\sigma}_d$		400.35	194.26	204.85	123.22	117.18	90.48	21.13	33.87	21.13	

Using formulas (3.8) and (3.11), the results for the sample data triangle are shown in Table 3.2. While formula (3.11) can be used to directly calculate the total variance of 2,447,095, segregating the covariance adjustment allows us to also directly calculate the total variance assuming zero correlation of 2,038,397. The coefficient of variation (CoV) column is the standard deviation divided by the mean.

Table 3.2 – Mack Estimated Unpaid Claims and Standard Deviations

		$\hat{R}(w)$	$\sqrt{Var[\hat{R}(w)]}$	CoV	CVA	$\sqrt{Var[\hat{R}(w)']}$	CoV
w	1	-	-	0.0%	-	-	0.0%
	2	94,634	75,535	79.8%	-	75,535	79.8%
	3	469,511	121,699	25.9%	*81,086*	146,238	31.1%
	4	709,638	133,549	18.8%	*139,674*	193,246	27.2%
	5	984,889	261,406	26.5%	*176,876*	315,624	32.0%
	6	1,419,459	411,010	29.0%	*259,674*	486,168	34.3%
	7	2,177,641	558,317	25.6%	*388,850*	680,384	31.2%
	8	3,920,301	875,328	22.3%	*573,313*	1,046,368	26.7%
	9	4,278,972	971,258	22.7%	*721,693*	1,210,034	28.3%
	10	4,625,811	1,363,155	29.5%	*841,236*	1,601,833	34.6%
CVA			*1,353,961*		*1,353,961*		
Total		**18,680,856**	**2,447,095**	**13.1%**		**2,447,095**	**13.1%**
Ex CVA			*2,038,397*	*10.9%*			

In addition to the commonly used display of the Mack estimates in the first three columns of Table 3.2, an interesting alternative is to include the covariance adjustment with the accident years. Since the covariance adjustment in formula (3.11) includes portions related to each accident year, we can include the portion related to each accident year in an expansion of formula (3.8).

$$Var\left[\hat{R}(w)'\right] = \hat{c}(w,n)^2 \times \sum_{d=n+1-w}^{n-1} \frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \left\{\frac{1}{\hat{c}(w,d)} + \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right\} + 2\hat{c}(w,n)\left(\sum_{i=w+1}^{N} c(i,n)\right) \sum_{d=n+1-w}^{n-1} \left(\frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right) \tag{3.12}$$

When using formula (3.12) to include a portion of the covariance adjustment, the formula for the total variance shown in (3.10) is revised as shown in formula (3.13).

$$Var\left[\hat{R}(tot)\right] = Var\left[\hat{R}(2)'\right] + Var\left[\hat{R}(3)'\right] + \cdots + Var\left[\hat{R}(N)'\right] \tag{3.13}$$

This alternative view of the Mack estimates is also included in Table 3.2, starting with the column that shows the portion of the covariance adjustment "allocated" to each accident year.[8] Note that for the alternative view the CoVs exhibit a smoother transition from the oldest year to the most current year, which may make comparisons to other models more consistent.

3.4 Unpaid Claim Runoff Uncertainty

Before looking at the formulas for the time horizon calculations introduced by Merz-Wüthrich, it is useful to start with the runoff of the unpaid claims.[9] The Mack runoff formulas can be used to calculate the risk margin using the cost of capital method and it will be a useful comparison to the runoff using the Merz-Wüthrich formulas.

In order to extend the Mack formulas for the runoff of the unpaid claims, we must first review the notation related to time. For this purpose we will designate the "at time" using a subscript $t = 0,1,\dots,u$ and we will designate the "time-horizon" within the formulas using a superscript $T = 1,2,\dots,U$. Including this new notation, we could restate the results from formulas (3.8) and (3.11) as $Var[\hat{R}_0^U(w)]$ and $Var[\hat{R}_0^U(tot)]$, respectively.[10] In this case, since we are starting from the end of the known data, D, the subscript is zero and because both the process and parameter variances are being calculated over the entire time horizon this is

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[8] Technically, the CVA column is only the covariance portion of formula (3.12) and the alternative standard deviation column can be calculated from the square root of the sum of the squares of the original standard deviation column and the CVA column.

[9] As noted in Section 1.1, proofs for the original Mack formulas are not included with this paper and as the extensions in Sections 3.4 and 3.5 follow the same assumptions and formulations, although reorganized for the cash flows, they are included without proofs.

[10] Both the subscripts and superscripts are shown here as a bridge to the time horizon discussion that starts in section 4, but for any formula where the subscript is absent it can be assumed to be zero and when the superscript is absent it can be assumed to be U.

commonly referred to as the "ultimate" time horizon, which is designated with the superscript U.

If we start by running off the estimated unpaid claims, the notation in section 2 can be restated for $t = 1$ as shown in formula (3.14) for $w = 3,4, \ldots, N$.

$$\hat{R}_1(w) = \hat{U}(w) - \hat{c}(w, N - w + 2) \tag{3.14}$$

We can generalize this further for any t as shown in formula (3.15) for $w = t + 2, t + 3, \ldots, N$.

$$\hat{R}_t(w) = \hat{U}(w) - \hat{c}(w, N - w + t + 1) \tag{3.15}$$

Applying formulas (3.14) and (3.15) to the sample data we can show the runoff of the estimated unpaid claims in Table 3.3.

Table 3.3 – Runoff of Estimated Unpaid Claims

	$\hat{R}_t(w)$								
w \ t =	0	1	2	3	4	5	6	7	8
1	-	-	-	-	-	-	-	-	-
2	94,634	-	-	-	-	-	-	-	-
3	469,511	93,678	-	-	-	-	-	-	-
4	709,638	462,448	92,268	-	-	-	-	-	-
5	984,889	650,741	424,066	84,611	-	-	-	-	-
6	1,419,459	1,036,173	684,625	446,148	89,016	-	-	-	-
7	2,177,641	1,572,093	1,147,592	758,242	494,122	98,588	-	-	-
8	3,920,301	2,610,043	1,884,254	1,375,463	908,802	592,237	118,164	-	-
9	4,278,972	3,260,138	2,170,522	1,566,954	1,143,840	755,764	492,507	98,266	-
10	4,625,811	3,769,007	2,871,597	1,911,841	1,380,205	1,007,518	665,692	433,810	86,555
Total	**18,680,856**	**13,454,320**	**9,274,925**	**6,143,258**	**4,015,986**	**2,454,107**	**1,276,363**	**532,076**	**86,555**

Running off the variance of the unpaid claims by accident year for $t = 1$, formula (3.8) can be restated as formula (3.16) for $w = 3,4, \ldots, N$.

$$Var\left[\hat{R}_1(w)\right] = \hat{c}(w,n)^2 \times \sum_{d=n+2-w}^{n-1} \frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \left\{\frac{1}{\hat{c}(w,d)} + \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right\} \tag{3.16}$$

Generalizing this further for any t is shown in formula (3.17) for $w = t + 2, t + 3, \ldots, N$.

$$Var\left[\hat{R}_t(w)\right] = \hat{c}(w,n)^2 \times \sum_{d=n+t+1-w}^{n-1} \frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \left\{\frac{1}{\hat{c}(w,d)} + \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right\} \tag{3.17}$$

Similarly, running off the total variance of the unpaid claims for all accident years for $t = 1$, formula (3.11) can be restated as formula (3.18) for $w = 3,4, \ldots, N$.

$$Var\left[\hat{R}_1(tot)\right] = \sum_{w=3}^{N} \left\{ Var\left[\hat{R}_1(w)\right] + 2\hat{c}(w,n) \times \left(\sum_{i=w+1}^{N} c(i,n) \right) \sum_{d=n+1-w}^{n-1} \left(\frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)} \right) \right\} \quad (3.18)$$

Generalizing this further for any t is shown in formula (3.19) for $w = t+2, t+3, \ldots, N$.

$$Var\left[\hat{R}_t(tot)\right] = \sum_{w=t+2}^{N} \left\{ Var\left[\hat{R}_t(w)\right] + 2\hat{c}(w,n) \times \left(\sum_{i=w+1}^{N} c(i,n) \right) \sum_{d=n+1-w}^{n-1} \left(\frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)} \right) \right\} \quad (3.19)$$

Applying formulas (3.16) to (3.19) to the sample data, we can show the runoff of the estimated standard deviations of the unpaid claims in Table 3.4.

Table 3.4 – Runoff of Estimated Standard Deviations of the Unpaid Claims

	$\sqrt{Var[\hat{R}_t(w)]}$								
t =	0	1	2	3	4	5	6	7	8
w 1	-	-	-	-	-	-	-	-	-
2	75,535	-	-	-	-	-	-	-	-
3	121,699	74,931	-	-	-	-	-	-	-
4	133,549	120,373	74,041	-	-	-	-	-	-
5	261,406	125,695	113,131	69,186	-	-	-	-	-
6	411,010	269,797	130,224	117,306	71,982	-	-	-	-
7	558,317	437,273	287,714	139,969	126,301	78,029	-	-	-
8	875,328	623,100	489,142	323,291	159,581	144,441	90,307	-	-
9	971,258	785,070	557,224	436,400	287,117	139,643	125,999	77,826	-
10	1,363,155	903,373	729,436	516,796	404,139	265,121	127,697	114,976	70,421
CVA	*1,353,961*	*1,039,055*	*773,477*	*556,945*	*384,712*	*263,965*	*170,358*	*79,424*	-
Total	**2,447,095**	**1,788,912**	**1,340,940**	**954,131**	**663,602**	**431,762**	**263,362**	**159,952**	**70,421**

As expected, the standard deviations decrease in a similar fashion to the estimated unpaid claims and when $t = 8$ there is no longer a covariance adjustment term since there is only one "cell" remaining. As another test of the entire runoff process, we can look at the coefficients of variation shown in Table 3.5.

Table 3.5 – Runoff of Coefficients of Variation of the Unpaid Claims

CoV

w	t =	0	1	2	3	4	5	6	7	8
w	1	-	-	-	-	-	-	-	-	-
	2	79.8%	-	-	-	-	-	-	-	-
	3	25.9%	80.0%	-	-	-	-	-	-	-
	4	18.8%	26.0%	80.2%	-	-	-	-	-	-
	5	26.5%	19.3%	26.7%	81.8%	-	-	-	-	-
	6	29.0%	26.0%	19.0%	26.3%	80.9%	-	-	-	-
	7	25.6%	27.8%	25.1%	18.5%	25.6%	79.1%	-	-	-
	8	22.3%	23.9%	26.0%	23.5%	17.6%	24.4%	76.4%	-	-
	9	22.7%	24.1%	25.7%	27.9%	25.1%	18.5%	25.6%	79.2%	-
	10	29.5%	24.0%	25.4%	27.0%	29.3%	26.3%	19.2%	26.5%	81.4%
	Total	**13.1%**	**13.3%**	**14.5%**	**15.5%**	**16.5%**	**17.6%**	**20.6%**	**30.1%**	**81.4%**

From Table 3.5 we can see that the total coefficient of variation increases as we progress from $t = 0,1,\ldots,8$. This makes sense statistically as estimates further in the future should be relatively more uncertain.

Adjusting the generalized formula (3.17) to include the covariance adjustment related to each accident year, we can use formula (3.20).

$$Var\left[\hat{R}_t(w)'\right] = \hat{c}(w,n)^2 \times \sum_{d=n+t+1-w}^{n-1} \frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \left\{\frac{1}{\hat{c}(w,d)} + \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right\} + 2\hat{c}(w,n) \times \left(\sum_{i=w+1}^{N} c(i,n)\right) \sum_{d=n+1-w}^{n-1} \left(\frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right) \tag{3.20}$$

Using formula (3.20), the runoff of the standard deviations in Table 3.4 can be restated as shown in Table 3.6.

Table 3.6 – Runoff of Estimated Standard Deviations of the Unpaid Claims

$\sqrt{Var[\hat{R}_t(w)']}$

w	t =	0	1	2	3	4	5	6	7	8
w	1	-	-	-	-	-	-	-	-	-
	2	75,535	-	-	-	-	-	-	-	-
	3	146,238	74,931	-	-	-	-	-	-	-
	4	193,246	144,569	74,041	-	-	-	-	-	-
	5	315,624	182,890	136,340	69,186	-	-	-	-	-
	6	486,168	322,928	185,489	139,093	71,982	-	-	-	-
	7	680,384	516,048	342,289	197,511	149,869	78,029	-	-	-
	8	1,046,368	761,474	577,804	384,423	225,461	171,765	90,307	-	-
	9	1,210,034	960,541	700,295	528,807	351,362	210,222	156,485	77,826	-
	10	1,601,833	1,125,689	893,426	647,922	488,300	326,547	191,615	139,742	70,421
	Total	**2,447,095**	**1,788,912**	**1,340,940**	**954,131**	**663,602**	**431,762**	**263,362**	**159,952**	**70,421**

The coefficients of variation comparing the standard deviations in Table 3.6 to the expected values in Table 3.3 are shown in Table 3.7. As noted above for Table 3.2, there is a smoother transition of all CoVs from the oldest year to the most current year.

Table 3.7 – Runoff of Coefficients of Variation of the Unpaid Claims

	CoV								
t =	0	1	2	3	4	5	6	7	8
w 1	-	-	-	-	-	-	-	-	-
2	79.8%	-	-	-	-	-	-	-	-
3	31.1%	80.0%	-	-	-	-	-	-	-
4	27.2%	31.3%	80.2%	-	-	-	-	-	-
5	32.0%	28.1%	32.2%	81.8%	-	-	-	-	-
6	34.3%	31.2%	27.1%	31.2%	80.9%	-	-	-	-
7	31.2%	32.8%	29.8%	26.0%	30.3%	79.1%	-	-	-
8	26.7%	29.2%	30.7%	27.9%	24.8%	29.0%	76.4%	-	-
9	28.3%	29.5%	32.3%	33.7%	30.7%	27.8%	31.8%	79.2%	-
10	34.6%	29.9%	31.1%	33.9%	35.4%	32.4%	28.8%	32.2%	81.4%
Total	**13.1%**	**13.3%**	**14.5%**	**15.5%**	**16.5%**	**17.6%**	**20.6%**	**30.1%**	**81.4%**

3.5 Cash Flow Uncertainty

In order to extend the Mack formulas for the uncertainty of the cash flows we need to focus on the calendar year diagonals where $k = w - d + 1$. Starting with the calendar year estimated unpaid claims, we can introduce new notation for cash flow, $CF(k)$, and use the formula shown in formula (3.21) for $k = N + 1, N + 2, \dots, N + n$.

$$\widehat{CF}(k) = \sum_{j=k-N}^{N} \begin{Bmatrix} \hat{c}(j, N-j+2) - c(j, N-j+1); & k = N+1 \\ \hat{c}(j, N-j+2) - \hat{c}(j, N-j+1); & k > N+1 \end{Bmatrix} \tag{3.21}$$

Of course, summing the estimated unpaid for all calendar years as shown in formula (3.22) should result in the same total estimated unpaid as in formula (3.9).

$$\widehat{CF}(tot) = \widehat{CF}(N+1) + \widehat{CF}(N+2) + \cdots + \widehat{CF}(N+n) \tag{3.22}$$

Reorganizing Mack's formula (3.8) for the variance of each accident year into a diagonal sum results in formula (3.23). Note, however, that while the variance for an accident year is based on the ultimate estimated amount for that accident year, for the calendar year each of the accident year component variances are based on the estimated cumulative amount for next year end, i.e., formula (3.23) uses $\hat{c}(j,N\text{-}j\text{+}2)^2$ instead of $\hat{c}(j,n)^2$.

$$Var\left[\widehat{CF}(k)\right] = \sum_{j=k-N}^{N} \hat{c}(j,N\text{-}j\text{+}2)^2 \times \frac{\hat{\sigma}_{N-j+1}^2}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \frac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \frac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} \tag{3.23}$$

Similar to formula (3.10), the total variance for all calendar years is the sum of the variances plus 2 times the covariance, as shown in formula (3.24).

$$Var\big[\widehat{CF}(tot)\big] = Var\big[\widehat{CF}(N+1)\big] + Var\big[\widehat{CF}(N+2)\big] + \cdots + Var\big[\widehat{CF}(N+n)\big] \\ +2 \times CoVariance \tag{3.24}$$

Similarly, formula (3.11) for the variance of the total of all accident years can be reorganized as the sum of the calendar years, plus the differences between the accident year variances from formula (3.8) and calendar year variances from formula (3.23), as shown in formula (3.25).

$$\begin{aligned} Var\big[\widehat{CF}(tot)\big] = \sum_{k=N+1}^{N+n} \Bigg\{ & Var\big[\widehat{CF}(k)\big] + 2\hat{c}(k-N+1,n) \\ & \times \left(\sum_{i=k-N+2}^{N} c(i,n)\right) \sum_{d=N+n-k}^{n-1} \left(\frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right)\Bigg\} \\ & + \sum_{j=k-N}^{N} \left[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2\right] \times \frac{\hat{\sigma}_{N-j+1}^2}{\hat{F}(N\text{-}j\text{+}1)^2} \\ & \times \left\{\frac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \frac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)}\right\} \end{aligned} \tag{3.25}$$

Table 3.8 – Mack Estimated Cash Flows and Standard Deviations

		$\widehat{CF}(k)$	$\sqrt{Var[\widehat{CF}(k)]}$	CoV	CVA	$\sqrt{Var[\widehat{CF}(k)']}$	CoV
k	11	5,226,536	665,562	12.7%	*1,531,370*	1,669,750	31.9%
	12	4,179,394	609,716	14.6%	*1,015,053*	1,184,097	28.3%
	13	3,131,668	558,467	17.8%	*758,861*	942,208	30.1%
	14	2,127,272	445,167	20.9%	*521,368*	685,565	32.2%
	15	1,561,879	353,389	22.6%	*359,256*	503,933	32.3%
	16	1,177,744	248,729	21.1%	*234,931*	342,139	29.1%
	17	744,287	142,151	19.1%	*153,519*	209,224	28.1%
	18	445,521	118,457	26.6%	*81,200*	143,616	32.2%
	19	86,555	70,421	81.4%	-	70,421	81.4%
	CVA		*2,106,547*		*2,106,547*		
	Total	**18,680,856**	**2,447,095**	**13.1%**		**2,447,095**	**13.1%**

As with other modeling frameworks, the sums of the means and variances by diagonal should be consistent with the sums by row, as seen in Table 3.8. In other words, the totals in the first three columns of Table 3.8 are the same as the totals in the first three columns of Table 3.2. Ideally, the CoVs should increase steadily as the future diagonals should represent more uncertainty, i.e., as k increases from 11 to 19, but for the data in the example the CoVs are relatively consistent from $k = 11, \ldots, 18$ and then jump significantly for $k = 19$.

Similar to the adjustment of the accident year variance in formula (3.12), the expansion to formula (3.23) to include a portion of the covariance adjustment by calendar year is shown as formula (3.26). The alternative view the CoVs in Table 3.8 exhibit a similar consistency from $k = 11, \ldots, 18$ and then jump significantly for $k = 19$. Note that the covariance adjustment excludes the last diagonal, i.e., $i = w + N - 1$ in formula (3.11), so none of the CVA is allocated to $k = 19$ in Table 3.8.

$$Var\left[\widehat{CF}(k)'\right] = \sum_{j=k-N}^{N} \hat{c}(j,N\text{-}j\text{+}2)^2 \times \frac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{\frac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \frac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)}\right\}$$
$$+2\hat{c}(k-N+1,n) \times \left(\sum_{i=k-N+2}^{N} c(i,n)\right) \sum_{d=N+n-k}^{n-1} \left(\frac{\hat{\sigma}^2_d}{\hat{F}(d)^2} \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right) \tag{3.26}$$
$$+[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2] \times \frac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{\frac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \frac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)}\right\}$$

After revising formula (3.23) to include a portion of the covariance adjustment, formula (3.24) for the total variance is revised as shown in formula (3.27).

$$Var\left[\widehat{CF}(tot)\right] = Var\left[\widehat{CF}(N+1)'\right] + Var\left[\widehat{CF}(N+2)'\right] + \cdots + Var\left[\widehat{CF}(N+n)'\right] \tag{3.27}$$

4. MERZ & WÜTHRICH MODEL

The premise of the 1-year time horizon is that if claims develop unfavorably over the subsequent 12 months and capital becomes impaired then management could intervene. Based on this premise as implemented for the Solvency II regime, the Merz & Wüthrich model calculates the uncertainty in the reserves after one year given the total uncertainty (i.e., the possible outcomes) during the first year. In other words, over a 1-year time horizon (i.e., the first diagonal), all possible outcomes should be considered and then the new reserves, conditional on each possible outcome, are calculated.

4.1 Uncertainty by Accident Year: One-Year Time Horizon

The formulas developed by Merz & Wüthrich [7] to calculate the unpaid claim uncertainty over a 1-year time horizon build on Mack's formulas and assumptions shown in (3.1) to (3.7). Starting with Mack's accident year uncertainty from (3.8), Merz-Wüthrich split the formula into components based on the first diagonal and the remaining diagonals as shown in (4.1).

$$Var\left[\hat{R}^1(w)\right] = \hat{c}(w,n)^2 \times \frac{\hat{\sigma}^2_{N+1-w}}{\hat{F}(N+1-w)^2} \times \left\{\frac{1}{\hat{c}(w,N+1-w)} + \frac{1}{\sum_{j=1}^{w-1} c(j,N+1-w)}\right\}$$
$$+\hat{c}(w,n)^2 \times \sum_{d=n+2-w}^{n-1} \frac{\hat{\sigma}^2_d}{\hat{F}(d)^2} \times \left\{\alpha^1_d \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right\} \tag{4.1}$$

For the first diagonal, both the process and parameter uncertainty are included such that the results will exactly match the Mack results for the first diagonal as in formula (3.23). For the remaining diagonals, only the parameter uncertainty is included and it is also reduced a bit using a weight function, α^1_d, which is calculated using formula (4.2).

$$\alpha^1_d = \frac{c(N+1-d,d)}{\sum_{j=1}^{N+1-d} c(j,d)}; for\ d = 1,2,\dots,N \tag{4.2}$$

The use of color for the weight components of the formulas is useful for clarifying the calculations and tracing the components through the various formulas.[11] The weight function can be thought of as an adjustment to the development factor, $F(d)$, and the parameter uncertainty for the years after the time horizon.

4.2 Total Uncertainty: One-Year Time Horizon

Adjusting the Mack formula (3.11) for the total uncertainty for the 1-year time horizon, Merz-Wüthrich developed formula (4.3), which also separates the covariance into the first diagonal and remaining diagonal components.

$$Var\left[\hat{R}^1(tot)\right] = \sum_{w=2}^{N}\left\{Var\left[\hat{R}^1(w)\right] + 2\hat{c}(w,n) \times \left(\sum_{i=w+1}^{N} c(i,n)\right)\right.$$
$$\times \left[\frac{\hat{\sigma}^2_{N+1-w}}{\hat{F}(N+1-w)^2} \times \frac{1}{\sum_{j=1}^{w-1} c(j,N+1-w)}\right.$$
$$\left.\left. + \sum_{d=n+2-w}^{n-1}\left(\frac{\hat{\sigma}^2_d}{\hat{F}(d)^2} \times \alpha^1_d \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right)\right]\right\} \tag{4.3}$$

Using formulas (4.1) and (4.3), the results for the sample data triangle are shown in Table 4.1. Comparing the results in the first three columns of Table 4.1 with the same columns in Table 3.2, note that for $w = 2$ the uncertainty is entirely for the first diagonal and, as such, the

[11] Alternatively, the weight functions could be colored as part of the parameter uncertainty, but using a different color will help in later parts of the paper.

standard deviations are exactly the same.[12] For $w > 2$ the uncertainties in Table 4.1 are a combination of the first diagonal and the remaining diagonals and, as such, the standard deviations are less than those in Table 3.2. Finally, the covariance adjustment for the total uncertainty in Table 4.1 is also less than in Table 3.2, resulting in a total standard deviation of 1,778,968 compared to 2,447,095.

Table 4.1 – Merz-Wüthrich Estimated Unpaid Claims and Standard Deviations

		$\hat{R}^1(w)$	$\sqrt{Var[\hat{R}^1(w)]}$	CoV	CVA	$\sqrt{Var[\hat{R}^1(w)']}$	CoV
w	1	-	-	0.0%	-	-	0.0%
	2	94,634	75,535	79.8%	-	75,535	79.8%
	3	469,511	105,309	22.4%	*81,086*	132,910	28.3%
	4	709,638	79,846	11.3%	*129,729*	152,332	21.5%
	5	984,889	235,115	23.9%	*150,379*	279,093	28.3%
	6	1,419,459	318,427	22.4%	*226,186*	390,584	27.5%
	7	2,177,641	361,089	16.6%	*323,435*	484,763	22.3%
	8	3,920,301	629,681	16.1%	*441,515*	769,047	19.6%
	9	4,278,972	588,662	13.8%	*541,749*	800,010	18.7%
	10	4,625,811	1,029,925	22.3%	*600,426*	1,192,165	25.8%
	CVA		*1,025,050*		*1,025,050*		
	Total	**18,680,856**	**1,778,968**	**9.5%**		**1,778,968**	**9.5%**
	Ex CVA		*1,453,959*	*7.8%*			

Similar to the alternative view of the covariance adjustment by accident year for the Mack model, a portion of the covariance adjustment in formula (4.3) can be included with formula (4.1) as shown in formula (4.4).

[12] As the Merz-Wüthrich formulas only address changes to the Mack standard deviations, the expected values are the same – i.e., $\hat{R}(w) = \hat{R}^1(w)$. The identical standard deviations for both Mack and Merz-Wüthrich for $w = 2$ is expected since the first diagonal includes both process and parameter variance for both formulas.

$$Var\left[\hat{R}^1(w)'\right] = \hat{c}(w,n)^2 \times \frac{\hat{\sigma}^2_{N+1-w}}{\hat{F}(N{+}1{-}w)^2} \times \left\{\frac{1}{\hat{c}(w,N{+}1{-}w)} + \frac{1}{\sum_{j=1}^{w-1} c(j,N{+}1{-}w)}\right\}$$
$$+\hat{c}(w,n)^2 \times \sum_{d=n+2-w}^{n-1} \frac{\hat{\sigma}^2_d}{\hat{F}(d)^2} \times \left\{\alpha^1_d \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right\}$$
$$+2\hat{c}(w,n) \times \left(\sum_{i=w+1}^{N} c(i,n)\right) \tag{4.4}$$
$$\times \left[\frac{\hat{\sigma}^2_{N+1-w}}{\hat{F}(N+1-w)^2} \times \frac{1}{\sum_{j=1}^{w-1} c(j,N+1-w)} + \sum_{d=n+2-w}^{n-1} \left(\frac{\hat{\sigma}^2_d}{\hat{F}(d)^2} \times \alpha^1_d \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right)\right]$$

This alternative view of the Merz-Wüthrich estimates is also included in Table 4.1, starting with the column that shows the portion of the covariance adjustment "allocated" to each accident year. Note that for the alternative view the CoVs exhibit a smoother transition from the oldest year to the most current year similar to the Mack alternative view.

4.3 Uncertainty by Accident Year: T-Year Time Horizon

The formulas developed by Merz & Wüthrich [7] above were subsequently extended in Merz & Wüthrich [8] to runoff the unpaid claim estimates for later time windows. Starting with $T' = 2$, formula (4.1) is extended as shown in formula (4.5). In the Merz & Wüthrich [8] paper, the authors describe extensions of the "time horizon" for $T > 1$, but since the first diagonal in formula (4.5) does not include all of the process and parameter variances, in this paper we will refer to the extensions as "time windows" (and use the T' notation) to improve clarity between models.[13]

$$Var\left[\hat{R}^2(w)\right] =$$
$$\hat{c}(w,n)^2 \times \frac{\hat{\sigma}^2_{N+2-w}}{\hat{F}(N{+}2{-}w)^2} \times \left\{\frac{1}{\hat{c}(w,N{+}2{-}w)} + (1-\alpha^1_{N+2-w}) \times \frac{1}{\sum_{j=1}^{w-1} c(j,N{+}2{-}w)}\right\}$$
$$+\hat{c}(w,n)^2 \times \sum_{d=n+3-w}^{n-1} \frac{\hat{\sigma}^2_d}{\hat{F}(d)^2} \times \left\{\alpha^2_d \times (1-\alpha^1_d) \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right\} \tag{4.5}$$

Note that in the extension for $T' = 2$, one minus the weights for $T = 1$ are used and the formula for the weights for $T' = 2$ are as in formula (4.6). Note also that the calculation of the

[13] In some of the Tables that follow, the headers only refer to T' for simplicity but for $T = 1$ the conversion to T is implied.

weights includes estimated cumulative values when $T > 1$.

$$\alpha_d^2 = \frac{\hat{c}(N+2-d,d)}{\sum_{j=1}^{N+2-d} c(j,d)}; for\ d = 2,3,\ldots,N-1 \tag{4.6}$$

A further extension and generalization for $T' > 2$ is shown in formula (4.7).

$$\begin{aligned} &Var\left[\hat{R}^{T'}(w)\right] = \\ &\hat{c}(w,n)^2 \times \frac{\hat{\sigma}_{N+T-w}^2}{\hat{F}(N+T\text{-}w)^2} \times \left\{\frac{1}{\hat{c}(w,N+T\text{-}w)} + \prod_{m=1}^{T-1}(1-\alpha_{N+T-w}^m) \times \frac{1}{\sum_{j=1}^{w-1} c(j,N+T\text{-}w)}\right\} \\ &+\hat{c}(w,n)^2 \times \sum_{d=n+T+1-w}^{n-1} \frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \left\{\alpha_d^T \times \prod_{m=1}^{T-1}(1-\alpha_d^m) \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right\} \end{aligned} \tag{4.7}$$

And the extension for the weight function is shown in formula (4.8).

$$\alpha_d^T = \frac{\hat{c}(N+T-d,d)}{\sum_{j=1}^{N+T-d} c(j,d)}; for\ d = T, T+1,\ldots,N-T+1 \tag{4.8}$$

4.4 Total Uncertainty: T-Year Time Horizon

In Merz & Wüthrich [8] the extension of the total uncertainty for $T' = 2$ is shown in formula (4.9).

$$\begin{aligned} Var\left[\hat{R}^2(tot)\right] = \sum_{w=3}^{N} \Bigg\{ & Var\left[\hat{R}^2(w)\right] + 2\hat{c}(w,n) \times \left(\sum_{i=w+2}^{N} c(i,n)\right) \\ & \times \left[\frac{\hat{\sigma}_{N+2-w}^2}{\hat{F}(N+2-w)^2} \times (1-\alpha_{N+2-w}^1) \times \frac{1}{\sum_{j=1}^{w-1} c(j,N+2-w)} \right. \\ & \left. + \sum_{d=n+3-w}^{n-1} \left(\frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \alpha_d^2 \times (1-\alpha_d^1) \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right)\right]\Bigg\} \end{aligned} \tag{4.9}$$

The further extension and generalization for $T' > 2$ is shown in formula (4.10).

$$Var\left[\hat{R}^{T'}(tot)\right] = \sum_{w=T+1}^{N} \left\{ Var\left[\hat{R}^{T'}(w)\right] + 2\hat{c}(w,n)\left(\sum_{i=w+T}^{N} c(i,n)\right) \times \left[\frac{\hat{\sigma}^2_{N+T-w}}{\hat{F}(N+T-w)^2} \times \prod_{m=1}^{T-1}\left(1-\alpha^m_{N+T-w}\right) \times \frac{1}{\sum_{j=1}^{w-1} c(j,N+T-w)} + \sum_{d=n+T+1-w}^{n-1} \left(\frac{\hat{\sigma}^2_d}{\hat{F}(d)^2} \times \alpha^T_d \times \prod_{m=1}^{T-1}\left(1-\alpha^m_d\right) \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)} \right) \right] \right\} \tag{4.10}$$

Applying formulas (4.1) to (4.10) to the sample data results in the standard deviations by year as shown in Table 4.2, with the results from Table 4.1 repeated in the first column of Table 4.2.

Table 4.2 – Runoff of Merz-Wüthrich Estimated Standard Deviations of the Unpaid Claims

$\sqrt{Var[\hat{R}^{T'}(w)]}$

w	$T'=$	1	2	3	4	5	6	7	8	9	TOTAL
	1	-	-	-	-	-	-	-	-	-	-
	2	75,535	-	-	-	-	-	-	-	-	75,535
	3	105,309	60,996	-	-	-	-	-	-	-	121,699
	4	79,846	91,093	56,232	-	-	-	-	-	-	133,549
	5	235,115	60,577	82,068	51,474	-	-	-	-	-	261,406
	6	318,427	233,859	57,825	82,433	51,999	-	-	-	-	411,010
	7	361,089	328,989	243,412	59,162	85,998	54,343	-	-	-	558,317
	8	629,681	391,249	359,352	266,320	64,443	94,166	59,533	-	-	875,328
	9	588,662	554,574	344,763	318,493	236,576	56,543	83,645	52,965	-	971,258
	10	1,029,925	538,726	511,118	317,142	293,978	218,914	51,661	77,317	49,055	1,363,155
	CVA	*1,025,050*	*676,444*	*449,236*	*288,887*	*164,691*	*92,828*	*57,595*	*24,085*	-	*1,353,961*
	Total	**1,778,968**	**1,177,727**	**885,178**	**607,736**	**428,681**	**267,503**	**128,557**	**96,764**	**49,055**	**2,447,095**

As expected, the standard deviations decrease in a similar fashion to the estimated unpaid claims and when $T' = 9$ there is no covariance adjustment term since there is only one "cell" remaining. An additional part of the results in Table 4.2 is the Total column, which is the square root of the sum of the squares of the other columns. The Total column is an important result as the complete runoff from Merz-Wüthrich are intended to reconcile with the results from Mack. Comparing the Total column in Table 4.2 with the results in Table 3.2 we see that the estimates are identical.

The expected runoff of the unpaid claims for Merz-Wüthrich is identical to the runoff for Mack, as previously shown in Table 3.3. Dividing the standard deviations in Table 4.2 by the means in Table 3.3 results in the runoff of the coefficients of variation shown in Table 4.3.

Table 4.3 – Runoff of Merz-Wüthrich Coefficients of Variation of the Unpaid Claims

CoV										
w \ T' =	1	2	3	4	5	6	7	8	9	TOTAL
1	-	-	-	-	-	-	-	-	-	-
2	79.8%	-	-	-	-	-	-	-	-	79.8%
3	22.4%	65.1%	-	-	-	-	-	-	-	25.9%
4	11.3%	19.7%	60.9%	-	-	-	-	-	-	18.8%
5	23.9%	9.3%	19.4%	60.8%	-	-	-	-	-	26.5%
6	22.4%	22.6%	8.4%	18.5%	58.4%	-	-	-	-	29.0%
7	16.6%	20.9%	21.2%	7.8%	17.4%	55.1%	-	-	-	25.6%
8	16.1%	15.0%	19.1%	19.4%	7.1%	15.9%	50.4%	-	-	22.3%
9	13.8%	17.0%	15.9%	20.3%	20.7%	7.5%	17.0%	53.9%	-	22.7%
10	22.3%	14.3%	17.8%	16.6%	21.3%	21.7%	7.8%	17.8%	56.7%	29.5%
Total	**9.5%**	**8.8%**	**9.5%**	**9.9%**	**10.7%**	**10.9%**	**10.1%**	**18.2%**	**56.7%**	**13.1%**

Adjusting formulas (4.5) and (4.7) to include the covariance adjustment related to each accident year are left to the reader. Applying formula (4.4), and the extensions for formulas (4.5) and (4.7), the runoff of the standard deviations in Table 4.2 are restated in Table 4.4. The first column in Table 4.4 is from Table 4.1 but, more importantly, the Total column in Table 4.4 reconciles with the alternative view for Mack in Table 3.2.

Table 4.4 – Runoff of Merz-Wüthrich Estimated Standard Deviations of the Unpaid Claims

$\sqrt{Var[\hat{R}^{T'}(w)']}$										
w \ T' =	1	2	3	4	5	6	7	8	9	TOTAL
1	-	-	-	-	-	-	-	-	-	-
2	75,535	-	-	-	-	-	-	-	-	75,535
3	132,910	60,996	-	-	-	-	-	-	-	146,238
4	152,332	104,771	56,232	-	-	-	-	-	-	193,246
5	279,093	103,950	90,942	51,474	-	-	-	-	-	315,624
6	390,584	255,290	89,682	88,793	51,999	-	-	-	-	486,168
7	484,763	377,458	258,077	86,475	91,743	54,343	-	-	-	680,384
8	769,047	491,773	402,375	278,897	91,580	99,957	59,533	-	-	1,046,368
9	800,010	658,702	429,906	356,254	247,299	81,487	89,102	52,965	-	1,210,034
10	1,192,165	691,492	592,230	382,924	321,096	227,976	71,017	80,981	49,055	1,601,833
Total	**1,778,968**	**1,177,727**	**885,178**	**607,736**	**428,681**	**267,503**	**128,557**	**96,764**	**49,055**	**2,447,095**

Table 4.5 – Runoff of Merz-Wüthrich Coefficients of Variation of the Unpaid Claims

CoV										
w \ T' =	1	2	3	4	5	6	7	8	9	TOTAL
1	-	-	-	-	-	-	-	-	-	-
2	79.8%	-	-	-	-	-	-	-	-	79.8%
3	28.3%	65.1%	-	-	-	-	-	-	-	31.1%
4	21.5%	22.7%	60.9%	-	-	-	-	-	-	27.2%
5	28.3%	16.0%	21.4%	60.8%	-	-	-	-	-	32.0%
6	27.5%	24.6%	13.1%	19.9%	58.4%	-	-	-	-	34.3%
7	22.3%	24.0%	22.5%	11.4%	18.6%	55.1%	-	-	-	31.2%
8	19.6%	18.8%	21.4%	20.3%	10.1%	16.9%	50.4%	-	-	26.7%
9	18.7%	20.2%	19.8%	22.7%	21.6%	10.8%	18.1%	53.9%	-	28.3%
10	25.8%	18.3%	20.6%	20.0%	23.3%	22.6%	10.7%	18.7%	56.7%	34.6%
Total	**9.5%**	**8.8%**	**9.5%**	**9.9%**	**10.7%**	**10.9%**	**10.1%**	**18.2%**	**56.7%**	**13.1%**

The coefficients of variation comparing the standard deviations in Table 4.4 to the expected values in Table 3.3 are shown in Table 4.5. As noted above for Table 4.1, there is a smoother transition of all CoVs from the oldest year to the most current year.

4.5 Cash Flow Uncertainty

The calculation of the cash flow uncertainty under the time horizon view is more complicated than for the ultimate view using Mack. The extension of the Mack formulas to calculate the cash flow uncertainty only requires one set of formulas as shown in (3.23) and (3.25). The extension of the Merz-Wüthrich formulas to calculate the cash flow uncertainty depends on the length of the time window, resulting in a different set of formulas for each of $T = 1,2,\ldots,N-1$.[14]

Starting with the formulas for $T = 1$, it is more convenient to separate formula (3.23) into separate formulas for the first diagonal and remaining diagonals as shown in formula (4.11).

$$Var\left[\widehat{CF}^1(k)\right] = \sum_{j=k-N}^{N} \begin{cases} \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{\dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)}\right\} & ;k = N+1 \\ \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{\alpha^1_{N-j+1} \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)}\right\} & ;k > N+1 \end{cases} \tag{4.11}$$

For the total uncertainty, it is also more convenient to separate formula (3.26) into separate formulas for the first diagonal and remaining diagonals as shown in formula (4.12).

$$Var\left[\widehat{CF}^1(tot)\right]$$
$$= \sum_{k}^{N+n} \begin{cases} Var\left[\widehat{CF}^1(k)\right] + 2\hat{c}(k-N+1,n) \times \left(\sum\limits_{i=k-N+2}^{N} c(i,n)\right) \times \left[\dfrac{\hat{\sigma}^2_{k-N+1}}{\hat{F}(k-N+1)^2} \times \dfrac{1}{\sum_{j=1}^{w-1} c(j,k-N+1)}\right] & \\ \quad +\left[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2\right] \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{\dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)}\right\} & ;k = N+1 \\ Var\left[\widehat{CF}^1(k)\right] + 2\hat{c}(k-N+1,n) \times \left(\sum\limits_{i=k-N+2}^{N} c(i,n)\right) \times \left[\sum\limits_{d=k-N+2}^{n-1} \left(\dfrac{\hat{\sigma}^2_d}{\hat{F}(d)^2} \times \alpha^1_d \times \dfrac{1}{\sum_{j=1}^{N-d} c(j,d)}\right)\right] & \\ \quad +\left[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2\right] \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{\alpha^1_{N-j+1} + \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)}\right\} & ;k > N+1 \end{cases} \tag{4.12}$$

Using formulas (4.11) and (4.12), the results for the sample data triangle are shown in Table 4.6.

[14] Similar to the Mack extensions, the extensions for Merz-Wüthrich follow the assumptions and formulations of the original papers so they are included without proofs.

Table 4.6 – Estimated Merz-Wüthrich Cash Flow and Standard Deviations for T=1

		$\widehat{CF}^1(k)$	$\sqrt{Var[\widehat{CF}^1(k)]}$	CoV	CVA	$\sqrt{Var[\widehat{CF}^1(k)']}$	CoV
k	11	5,226,536	665,562	12.7%	*1,531,370*	1,669,750	31.9%
	12	4,179,394	111,733	2.7%	*348,793*	366,252	8.8%
	13	3,131,668	108,154	3.5%	*284,901*	304,739	9.7%
	14	2,127,272	95,702	4.5%	*226,334*	245,735	11.6%
	15	1,561,879	83,976	5.4%	*177,520*	196,381	12.6%
	16	1,177,744	76,031	6.5%	*141,832*	160,926	13.7%
	17	744,287	67,017	9.0%	*109,047*	127,994	17.2%
	18	445,521	55,652	12.5%	*60,893*	82,493	18.5%
	19	86,555	40,213	46.5%	-	40,213	46.5%
	CVA		*1,632,904*		*1,632,904*		
	Total	**18,680,856**	**1,778,968**	**9.5%**		**1,778,968**	**9.5%**

Comparing the first three columns in Table 4.6 with Table 3.8, it makes sense that for $k = 11$, i.e., the first diagonal, the results are identical and, comparing Table 4.6 with Table 4.1, the total results are also identical as expected.

Similar to the alternative view of the covariance adjustment by calendar year for the Mack model, a portion of the covariance adjustment in formula (4.12) can be included with formula (4.11) as shown in formula (4.13).

$$Var[\widehat{CF}^1(k)'] = \sum_{j=k-N}^{N} \left\{ \begin{array}{ll} \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} & \\ +2\hat{c}(k-N+1,n) \times \left(\sum_{i=k-N+2}^{N} c(i,n) \right) \left[\dfrac{\hat{\sigma}^2_{k-N+1}}{\hat{F}(k-N+1)^2} \times \dfrac{1}{\sum_{j=1}^{w-1} c(j,k-N+1)} \right] & ;k = N+1 \\ +[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2] \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} & \\ \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \alpha^1_{N-j+1} \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} & \\ +2\hat{c}(k-N+1,n) \times \left(\sum_{i=k-N+2}^{N} c(i,n) \right) \left[+ \sum_{d=k-N+2}^{n-1} \left(\dfrac{\hat{\sigma}^2_d}{\hat{F}(d)^2} \times \alpha^1_d \times \dfrac{1}{\sum_{j=1}^{N-d} c(j,d)} \right) \right] & ;k > N+1 \\ +[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2] \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \alpha^1_{N-j+1} + \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} & \end{array} \right\} \quad (4.13)$$

This alternative view of the Merz-Wüthrich cash flow estimates is also included in Table 4.6, starting with the column that shows the portion of the covariance adjustment "allocated" to each calendar year. Note that for the alternative view the CoVs exhibit a smoother transition from the first diagonal to the last diagonal similar to the Mack alternative view.

Continuing with the formulas for $T' = 2$, the formulas for the first diagonal and remaining diagonals are shown in formula (4.14).

$$Var[\widehat{CF}^2(k)] = \sum_{j=k-N}^{N} \begin{Bmatrix} \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + (1-\alpha^1_{N-j+1}) \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} & ;k = N+2 \\ \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \alpha^2_{N-j+1} \times (1-\alpha^1_{N-j+1}) \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} & ;k > N+2 \end{Bmatrix} \quad (4.14)$$

For the total uncertainty, the formulas for the first diagonal and remaining diagonals are shown in formula (4.15).

$$Var[\widehat{CF}^2(tot)] = \sum_{k}^{N+n} \begin{Bmatrix} \begin{array}{c} Var[\widehat{CF}^2(k)] + 2\hat{c}(k-N+1,n) \times \left(\sum_{i=k-N+2}^{N} c(i,n) \right) \\ \times \left[\dfrac{\hat{\sigma}^2_{k-N+1}}{\hat{F}(k-N+1)^2} \times (1-\alpha^1_{k-N+1}) \times \dfrac{1}{\sum_{j=1}^{w-1} c(j,k-N+1)} \right] \\ +[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2] \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + (1-\alpha^1_{N-j+1}) \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} \end{array} & ;k = N+2 \\ \begin{array}{c} Var[\widehat{CF}^2(k)] + 2\hat{c}(k-N+1,n) \times \left(\sum_{i=k-N+2}^{N} c(i,n) \right) \\ \times \left[\sum_{d=k-N+2}^{n-1} \left(\dfrac{\hat{\sigma}^2_d}{\hat{F}(d)^2} \times \alpha^2_d \times (1-\alpha^1_d) \times \dfrac{1}{\sum_{j=1}^{N-d} c(j,d)} \right) \right] \\ +[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2] \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \alpha^2_{N-j+1} \times (1-\alpha^1_{N-j+1}) \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} \end{array} & ;k > N+2 \end{Bmatrix} \quad (4.15)$$

Using formulas (4.14) and (4.15), the results for the sample data triangle are shown in Table 4.7.

Table 4.7 – Estimated Merz-Wüthrich Cash Flow and Standard Deviations for T'=2

		$\widehat{CF}^2(k)$	$\sqrt{Var[\widehat{CF}^2(k)]}$	CoV	CVA	$\sqrt{Var[\widehat{CF}^2(k)']}$	CoV
k	11						
	12	4,179,394	599,391	14.3%	*953,245*	1,126,031	26.9%
	13	3,131,668	86,156	2.8%	*213,751*	230,461	7.4%
	14	2,127,272	76,066	3.6%	*161,420*	178,445	8.4%
	15	1,561,879	62,836	4.0%	*114,746*	130,825	8.4%
	16	1,177,744	51,412	4.4%	*82,900*	97,548	8.3%
	17	744,287	38,525	5.2%	*60,077*	71,368	9.6%
	18	445,521	31,819	7.1%	*31,311*	44,641	10.0%
	19	86,555	20,602	23.8%	-	20,602	23.8%
	CVA		*1,002,522*		*1,002,522*		
	Total	**13,454,320**	**1,177,727**	**8.8%**		**1,177,727**	**8.8%**

Comparing Table 4.7 with Table 3.8, note that the standard deviation for $k = 12$, i.e., the first diagonal at time $t = 1$, in Table 4.7 is less than in Table 3.8, which makes sense since formula (4.17) for the first diagonal only includes a portion of the parameter uncertainty. Comparing Table 4.7 with the columns for $T' = 2$ in Tables 4.2 and 4.3, the totals are identical as expected. Table 4.7 also includes the alternative view of the covariance adjustment, but the derivation of the formula is left to the reader.

Continuing with the formulas for $T' > 2$, the formulas for the first diagonal and remaining diagonals are shown in formula (4.16).

$$Var\left[\widehat{CF}^T(k)\right] = \sum_{j=k-N}^{N} \begin{cases} \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \prod_{m=1}^{T-1}\left(1-\alpha^m_{N-j+1}\right) \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} & ;k = N+T \\ \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \alpha^T_{N-j+1} \times \prod_{m=1}^{T-1}\left(1-\alpha^m_{N-j+1}\right) \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} & ;k > N+T \end{cases} \quad (4.16)$$

For the total uncertainty, the formulas for the first diagonal and remaining diagonals are shown in formula (4.17).

$$Var\left[\widehat{CF}^T(tot)\right] = \sum_{k}^{N+n} \begin{cases} Var\left[\widehat{CF}^2(k)\right] + 2\hat{c}(k-N+1,n) \times \left(\sum_{i=k-N+2}^{N} c(i,n)\right) \\ \quad \times \left[\dfrac{\hat{\sigma}^2_{k-N+1}}{\hat{F}(k-N+1)^2} \times \prod_{m=1}^{T-1}\left(1-\alpha^m_{N-j+1}\right) \times \dfrac{1}{\sum_{j=1}^{w-1} c(j,k-N+1)}\right] & ;k = N+T \\ \quad +\left[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2\right] \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \prod_{m=1}^{T-1}\left(1-\alpha^m_{N-j+1}\right) \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} & \\ Var\left[\widehat{CF}^2(k)\right] + 2\hat{c}(k-N+1,n) \times \left(\sum_{i=k-N+2}^{N} c(i,n)\right) \\ \quad \times \left[\sum_{d=k-N+2}^{n-1} \left(\dfrac{\hat{\sigma}^2_d}{\hat{F}(d)^2} \times \alpha^T_{N-j+1} \times \prod_{m=1}^{T-1}\left(1-\alpha^m_{N-j+1}\right) \times \dfrac{1}{\sum_{j=1}^{N-d} c(j,d)}\right)\right] & ;k > N+T \\ \quad +\left[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2\right] \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \alpha^T_{N-j+1} \times \prod_{m=1}^{T-1}\left(1-\alpha^m_{N-j+1}\right) \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} & \end{cases} \quad (4.17)$$

Using formulas (4.16) and (4.17), the results for the sample data triangle will be similar to the results shown in Table 4.7, meaning the first diagonal will be less than the same diagonal in Table 3.6 and the totals will match the same time window in Tables 4.2 and 4.3.

4.6 A Comparison of Mack vs. Merz-Wüthrich

Now that we have reviewed the various formulas related to the Mack and Merz-Wüthrich models, it is instructive to compare the runoff for the two models using the totals from Tables (3.3), (3.4), (3.5), (4.2), and (4.3). As shown in Table 4.8, at time $t = 0$ (and $T = 1$) the standard deviation for the 1-year time horizon is 72.7% of the standard deviation for the ultimate time horizon. As previously discussed, this makes sense since the 1-year time horizon only includes the parameter variance beyond the first diagonal.

Table 4.8 – Comparison of Estimated Runoff for Mack and Merz-Wüthrich Models

		$\hat{R}_t(tot)$	$\sqrt{Var[\hat{R}_t(tot)]}$	CoV	$\sqrt{Var[\hat{R}^{T\prime}(tot)]}$	CoV	$Ratio$
$t =$	0	18,680,856	2,447,095	13.1%	1,778,968	9.5%	72.7%
	1	13,454,320	1,788,912	13.3%	1,177,727	8.8%	65.8%
	2	9,274,925	1,340,940	14.5%	885,178	9.5%	66.0%
	3	6,143,258	954,131	15.5%	607,736	9.9%	63.7%
	4	4,015,986	663,602	16.5%	428,681	10.7%	64.6%
	5	2,454,107	431,762	17.6%	267,503	10.9%	62.0%
	6	1,276,363	263,362	20.6%	128,557	10.1%	48.8%
	7	532,076	159,952	30.1%	96,764	18.2%	60.5%
	8	86,555	70,421	81.4%	49,055	56.7%	69.7%

In the England, Verrall and Wüthrich [3] paper, the authors discuss using the runoff of the time window standard deviations for the runoff of the capital requirement in the cost of capital method of calculating the risk margin under Solvency II.[15] While the runoff of the time window standard deviations clearly reconcile[16] with the Mack standard deviations, it does not appear as though the runoff of the time window standard deviations adhere to the time horizon concept used for Solvency II. Thus, the Merz-Wüthrich would be more accurately described as a reasonable approximation for the runoff of the capital requirement.

To illustrate this issue, we start with $T = 1$ as shown in Table 4.6 and note that the first diagonal (i.e., for $k = 11$) is identical to the first diagonal in Table 3.8 since it includes both process and parameter uncertainty. The differences in the total uncertainty between Tables 4.6 and 3.8 is completely due to the remaining diagonals in Table 4.6 that only contain parameter uncertainty. This is the essence of the 1-year time horizon since the first diagonal should be an estimate of the total uncertainty and then we are concerned with estimating the change in reserves given the possible outcomes during the first year.

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[15] More specifically, the capital requirement is based on the 99.5th percentile of the 1-year time horizon unpaid claim distribution and the runoff of the capital requirement would be based on subsequent 99.5th percentiles as $T' = 1,2,\dots,N$.

[16] As shown in Table 4.2, the square root of the sum of the squares of the Merz-Wüthrich standard deviations by time window for each accident year and the total of all accident years are the same as the Mack standard deviations.

For the runoff of the time window, as we move to $T' = 2$ the same logic should continue to hold true, meaning after the first year is complete we would then want to estimate the total uncertainty for the next diagonal and the change in reserves given the possible outcomes during that second year. Following this logic, the first diagonal (i.e., for $k = 12$) in Table 4.7 should be identical to the second diagonal in Table 3.8. However, as seen in Table 4.7 the first diagonal is less than the second diagonal in Table 3.8 since it does not include all of the parameter uncertainty.

This issue can also be observed by comparing the oldest accident years for the runoff of the standard deviations in Tables 3.4 and 4.2. For example, in Table 3.4 the standard deviation for the oldest accident year when $t = 1$ is 74,931 and in Table 4.2 the standard deviation for the oldest accident year when $T' = 2$ is 60,996. Since both of these cells include only the first diagonal, the values should be the same. From the perspective of reconciling the runoff of Merz-Wüthrich with Mack this makes sense, but from the perspective of running off the required capital it does not make sense.

Another way to think about the runoff of the Merz-Wüthrich standard deviations is that they are always looking at the runoff from the perspective of the current time, or $t = 0$. From this perspective, in the second year (i.e., $T' = 2$) the first remaining diagonal (i.e., for $k = 12$ in Table 4.7) can be thought of as only containing enough uncertainty to reconcile with Mack at time $t = 0$. This perspective is also consistent with the total reserve notation in this section that does not contain a subscript implying that $t = 0$.

We can illustrate this issue with 2 other cases:

- If in the formula of α_d^T, $\hat{c}(N + T - d, d)$ is much greater than $\hat{c}(i, d)$ with $i < N + T - d$, then α_d^T tends to one and $(1 - \alpha_d^T)$ tends to 0. This would imply there is no more remaining parameter risk for $T' \geq 2$ linked to the diagonals $T' + 1$ which does not make sense in a run off of the required capital.
- We can also compare the case here with the re-reserving method or actuary in the box method (see Diers [2]). When simulating the T+1, ..., T+N diagonals with the Bootstrap incrementals, a full Chain Ladder is applied, i.e., the full estimation risk is calculated even if it was already partially captured in the previous diagonal run-off.

5. TIME-HORIZON UNCERTAINY: AN ALTERNATIVE APPROACH

In order to calculate the runoff of the required capital under Solvency II, we need to revise formulas (4.5), (4.7), (4,9), and (4.10) to include all of the parameter uncertainty for the first diagonals as the reserves runoff for $t > 0$.

5.1 Uncertainty by Accident Year: T-Year Time Horizon

Starting with $T = 2$, formula (4.5) must be revised as shown in formula (5.1), except that to clearly note that we are concerned with a 1-year time horizon one year in the future the notation has also been revised to show that $t = 1$ and $T = 1$.

$$Var\left[\hat{R}_1^1(w)\right] = \hat{c}(w,n)^2 \times \frac{\hat{\sigma}_{N+2-w}^2}{\hat{F}(N+2\text{-}w)^2} \times \left\{\frac{1}{\hat{c}(w,N+2\text{-}w)} + \frac{1}{\sum_{j=1}^{w-1} c(j,N+2\text{-}w)}\right\} + \hat{c}(w,n)^2 \times \sum_{d=n+3-w}^{n-1} \frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \left\{\alpha_d^2 \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right\} \quad (5.1)$$

Comparing formula (5.1) with formula (4.5), the one minus the weights for $T = 1$ portions have been removed, but the weights for $T = 2$, as in formula (4.6), is still included. This formula for the second year is consistent with formula (4.1) for the first year. The generalization for $t > 2$ is shown in formula (5.2).

$$Var\left[\hat{R}_t^1(w)\right] = \hat{c}(w,n)^2 \times \frac{\hat{\sigma}_{N+t+1-w}^2}{\hat{F}(N+t+1\text{-}w)^2} \times \left\{\frac{1}{\hat{c}(w,N+t+1\text{-}w)} + \frac{1}{\sum_{j=1}^{w-1} c(j,N+t+1\text{-}w)}\right\} + \hat{c}(w,n)^2 \times \sum_{d=n+t+2-w}^{n-1} \frac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \left\{\alpha_d^{t+1} \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right\} \quad (5.2)$$

The generalization for $t > 2$ uses the weights as shown in formula (4.8), except that $T = t + 1$ using the new notation with both subscripts and superscripts.

5.2 Total Uncertainty: T-Year Time Horizon

The revised formula for the total uncertainty when $t = 2$ is shown in formula (5.3).

$$Var\left[\hat{R}_1^1(tot)\right] = \sum_{w=3}^{N}\left\{Var\left[\hat{R}_1^1(w)\right] + 2\hat{c}(w,n)\left(\sum_{i=w+2}^{N} c(i,n)\right) \times \left[\frac{\hat{\sigma}^2_{N+2-w}}{\hat{F}(N+2-w)^2}\times\frac{1}{\sum_{j=1}^{w-1} c(j,N+2-w)} + \sum_{d=n+3-w}^{n-1}\left(\frac{\hat{\sigma}_d^2}{\hat{F}(d)^2}\times\alpha_d^2\times\frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right)\right]\right\} \tag{5.3}$$

The generalization for $t > 2$ is shown in formula (5.4). Comparing formulas (5.3) and (5.4) with formulas (4.9) and (4.10), respectively, the one minus the weights terms have been removed similar to formulas (5.1) and (5.2).

$$Var\left[\hat{R}_t^1(tot)\right] = \sum_{w=t+2}^{N}\left\{Var\left[\hat{R}_t^1(w)\right] + 2\hat{c}(w,n)\left(\sum_{i=w+t+1}^{N} c(i,n)\right) \times \left[\frac{\hat{\sigma}^2_{N+t+1-w}}{\hat{F}(N+t+1-w)^2}\times\frac{1}{\sum_{j=1}^{w-1} c(j,N+t+1-w)} + \sum_{d=n+t+2-w}^{n-1}\left(\frac{\hat{\sigma}_d^2}{\hat{F}(d)^2}\times\alpha_d^{t+1}\times\frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right)\right]\right\} \tag{5.4}$$

Applying formulas (5.1) to (5.4) to the sample data results in the standard deviations by year as shown in Table 5.1, with the results from Table 4.1 repeated in the first column of Table 5.1.

Table 5.1 – Runoff of Alternative Estimated Standard Deviations of the Unpaid Claims

		$\sqrt{Var[\hat{R}_t^1(w)]}$									
	t =	0	1	2	3	4	5	6	7	8	TOTAL
w	1	-	-	-	-	-	-	-	-	-	-
	2	75,535	-	-	-	-	-	-	-	-	75,535
	3	105,309	74,931	-	-	-	-	-	-	-	129,247
	4	79,846	100,806	74,041	-	-	-	-	-	-	148,389
	5	235,115	68,535	93,353	69,186	-	-	-	-	-	271,067
	6	318,427	240,563	67,590	95,673	71,982	-	-	-	-	422,102
	7	361,089	336,607	255,033	70,558	102,361	78,029	-	-	-	574,697
	8	629,681	400,731	374,947	284,965	79,593	116,320	90,307	-	-	898,273
	9	588,662	562,933	356,774	334,233	253,564	69,171	101,939	77,826	-	993,953
	10	1,029,925	544,418	521,865	329,305	308,794	234,466	62,194	92,663	70,421	1,380,457
	CVA	*1,025,050*	*787,105*	*592,464*	*434,573*	*299,857*	*212,772*	*154,021*	*79,424*	*-*	*1,541,216*
	Total	**1,778,968**	**1,258,989**	**987,439**	**713,534**	**521,112**	**353,057**	**214,796**	**144,746**	**70,421**	**2,588,861**

As expected, the standard deviations runoff in a similar fashion to the estimated unpaid claims and when $t = 8$ there is no covariance adjustment term since there is only one "cell" remaining. An additional part of the results in Table 5.1 is the Total column, which is the square root of the sum of the squares of the other columns. The Total column shows that this time horizon runoff does not reconcile with the results from Mack, but that is not the intent.

The expected runoff of the unpaid claims is identical to the runoff for Mack, as previously shown in Table 3.3. Dividing the standard deviations in Table 5.1 by the means in Table 3.3 results in the runoff of the coefficients of variation shown in Table 5.2.

Table 5.2 – Runoff of Alternative Coefficients of Variation of the Unpaid Claims

	CoV									
$t=$	0	1	2	3	4	5	6	7	8	TOTAL
w 1	-	-	-	-	-	-	-	-	-	-
2	79.8%	-	-	-	-	-	-	-	-	79.8%
3	22.4%	80.0%	-	-	-	-	-	-	-	27.5%
4	11.3%	21.8%	80.2%	-	-	-	-	-	-	20.9%
5	23.9%	10.5%	22.0%	81.8%	-	-	-	-	-	27.5%
6	22.4%	23.2%	9.9%	21.4%	80.9%	-	-	-	-	29.7%
7	16.6%	21.4%	22.2%	9.3%	20.7%	79.1%	-	-	-	26.4%
8	16.1%	15.4%	19.9%	20.7%	8.8%	19.6%	76.4%	-	-	22.9%
9	13.8%	17.3%	16.4%	21.3%	22.2%	9.2%	20.7%	79.2%	-	23.2%
10	22.3%	14.4%	18.2%	17.2%	22.4%	23.3%	9.3%	21.4%	81.4%	29.8%
Total	**9.5%**	**9.4%**	**10.6%**	**11.6%**	**13.0%**	**14.4%**	**16.8%**	**27.2%**	**81.4%**	**13.9%**

Adjusting formulas (5.1) and (5.2) to include the covariance adjustment related to each accident year are left to the reader. Applying formula (4.4), and the extensions for formulas (5.1) and (5.2), the runoff of the standard deviations in Table 5.1 are restated in Table 5.3.

Table 5.3 – Runoff of Alternative Estimated Standard Deviations of the Unpaid Claims

	$\sqrt{Var[\hat{R}_t^1(w)']}$									
$t=$	0	1	2	3	4	5	6	7	8	TOTAL
w 1	-	-	-	-	-	-	-	-	-	-
2	75,535	-	-	-	-	-	-	-	-	75,535
3	132,910	74,931	-	-	-	-	-	-	-	152,577
4	152,332	128,734	74,041	-	-	-	-	-	-	212,742
5	279,093	136,650	120,436	69,186	-	-	-	-	-	340,379
6	390,584	278,768	133,825	121,406	71,982	-	-	-	-	517,781
7	484,763	406,147	290,998	138,420	130,333	78,029	-	-	-	725,855
8	769,047	531,387	449,405	322,236	156,635	148,897	90,307	-	-	1,111,066
9	800,010	695,112	485,883	409,150	291,078	149,109	137,853	77,826	-	1,287,906
10	1,192,165	732,101	643,749	446,323	374,337	272,318	137,763	122,044	70,421	1,680,466
Total	**1,778,968**	**1,258,989**	**987,439**	**713,534**	**521,112**	**353,057**	**214,796**	**144,746**	**70,421**	**2,588,861**

The coefficients of variation comparing the standard deviations in Table 5.3 to the expected values in Table 3.3 are shown in Table 5.4. As noted above for Table 4.1, there is a smoother transition of all CoVs from the oldest year to the most current year.

Table 5.4 – Runoff of Alternative Coefficients of Variation of the Unpaid Claims

	CoV									
w \ t =	0	1	2	3	4	5	6	7	8	TOTAL
1	-	-	-	-	-	-	-	-	-	-
2	79.8%	-	-	-	-	-	-	-	-	79.8%
3	28.3%	80.0%	-	-	-	-	-	-	-	32.5%
4	21.5%	27.8%	80.2%	-	-	-	-	-	-	30.0%
5	28.3%	21.0%	28.4%	81.8%	-	-	-	-	-	34.6%
6	27.5%	26.9%	19.5%	27.2%	80.9%	-	-	-	-	36.5%
7	22.3%	25.8%	25.4%	18.3%	26.4%	79.1%	-	-	-	33.3%
8	19.6%	20.4%	23.9%	23.4%	17.2%	25.1%	76.4%	-	-	28.3%
9	18.7%	21.3%	22.4%	26.1%	25.4%	19.7%	28.0%	79.2%	-	30.1%
10	25.8%	19.4%	22.4%	23.3%	27.1%	27.0%	20.7%	28.1%	81.4%	36.3%
Total	**9.5%**	**9.4%**	**10.6%**	**11.6%**	**13.0%**	**14.4%**	**16.8%**	**27.2%**	**81.4%**	**13.9%**

5.3 Cash Flow Uncertainty

Similar to the accident year formulas, cash flow formulas (4.11), (4.12), and (4.13) do not need to be revised. Moving to the formulas for $t = 1$, the formulas for the first diagonal and remaining diagonals are shown in formula (5.5).

$$Var\left[\widehat{CF}_1^1(k)\right] = \sum_{j=k-N}^{N} \begin{cases} \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} & ; k = N+2 \\ \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \alpha^2_{N-j+1} \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} & ; k > N+2 \end{cases} \tag{5.5}$$

For the total uncertainty, the formulas for the first diagonal and remaining diagonals are shown in formula (5.6).

$$Var\left[\widehat{CF}_1^1(tot)\right] = \sum_{k}^{N+n} \begin{cases} \begin{aligned} & Var\left[\widehat{CF}^2(k)\right] + 2\hat{c}(k-N+1,n) \times \left(\sum_{i=k-N+2}^{N} c(i,n) \right) \\ & \times \left[\frac{\hat{\sigma}^2_{k-N+1}}{\hat{F}(k-N+1)^2} \times \frac{1}{\sum_{j=1}^{w-1} c(j,k-N+1)} \right] \\ & + \left[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2\right] \times \frac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \frac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \frac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} \end{aligned} & ; k = N+2 \\ \begin{aligned} & Var\left[\widehat{CF}^2(k)\right] + 2\hat{c}(k-N+1,n) \times \left(\sum_{i=k-N+2}^{N} c(i,n) \right) \\ & \times \left[\sum_{d=k-N+2}^{n-1} \left(\frac{\hat{\sigma}^2_d}{\hat{F}(d)^2} \times \alpha^2_d \times \frac{1}{\sum_{j=1}^{N-d} c(j,d)} \right) \right] \\ & + \left[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2\right] \times \frac{\hat{\sigma}^2_{N-j+1}}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \alpha^2_{N-j+1} \times \frac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} \end{aligned} & ; k > N+2 \end{cases} \tag{5.6}$$

Using formulas (5.5) and (5.6), the results for the sample data triangle are shown in Table 5.5.

Table 5.5 – Estimated Alternative Cash Flow and Standard Deviations for t=1

		$\widehat{CF}_1^1(k)$	$\sqrt{Var[\widehat{CF}_1^1(k)]}$	CoV	CVA	$\sqrt{Var[\widehat{CF}_1^1(k)']}$	CoV
k	11						
	12	4,179,394	609,716	14.6%	*1,015,053*	1,184,097	28.3%
	13	3,131,668	98,559	3.1%	*254,367*	272,794	8.7%
	14	2,127,272	87,848	4.1%	*197,935*	216,554	10.2%
	15	1,561,879	74,810	4.8%	*149,542*	167,210	10.7%
	16	1,177,744	64,972	5.5%	*115,815*	132,795	11.3%
	17	744,287	54,453	7.3%	*87,781*	103,298	13.9%
	18	445,521	45,194	10.1%	*48,293*	66,142	14.8%
	19	86,555	31,868	36.8%	-	31,868	36.8%
	CVA		*1,086,291*		*1,086,291*		
	Total	**13,454,320**	**1,258,989**	**9.4%**		**1,258,989**	**9.4%**

Comparing Table 5.5 with Table 3.8, note that the standard deviation for $k = 12$ in Table 5.5 is the same as in Table 3.6, which makes sense since formula (5.5) for the first diagonal includes all of the process and parameter uncertainty. Comparing Table 5.5 with the columns for $t = 1$ in Tables 5.1 and 5.2, the totals are identical as expected.

Similar to the alternative view of the covariance adjustment by calendar year for the Mack model, a portion of the covariance adjustment in formula (5.6) can be included with formula (5.5) as shown in formula (5.7).

$$Var[\widehat{CF}_1^1(k)'] = \sum_{j=k-N}^{N} \begin{cases} \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}_{N-j+1}^2}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} \\ +2\hat{c}(k-N+1,n) \times \left(\displaystyle\sum_{i=k-N+2}^{N} c(i,n) \right) \times \left[\dfrac{\hat{\sigma}_{k-N+1}^2}{\hat{F}(k-N+1)^2} \times \dfrac{1}{\sum_{j=1}^{w-1} c(j,k-N+1)} \right] & ;k = N+2 \\ +[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2] \times \dfrac{\hat{\sigma}_{N-j+1}^2}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} \\ \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}_{N-j+1}^2}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \alpha_{N-j+1}^2 \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} \\ +2\hat{c}(k-N+1,n) \times \left(\displaystyle\sum_{i=k-N+2}^{N} c(i,n) \right) \times \left[\displaystyle\sum_{d=k-N+2}^{n-1} \left(\dfrac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \alpha_d^2 \times \dfrac{1}{\sum_{j=1}^{N-d} c(j,d)} \right) \right] & ;k > N+2 \\ +[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2] \times \dfrac{\hat{\sigma}_{N-j+1}^2}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{ \alpha_{N-j+1}^2 \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)} \right\} \end{cases} \quad (5.7)$$

This alternative view of the alternative cash flow estimates is also included in Table 5.5, starting with the column that shows the portion of the covariance adjustment "allocated" to each calendar year. Note that for the alternative view the CoVs exhibit a smoother transition from the first diagonal to the last diagonal similar to the Mack alternative view.

Continuing with the formulas for $t > 2$, the formulas for the first diagonal and remaining diagonals are shown in formula (5.8).

$$Var\left[\widehat{CF}_t^1(k)\right] = \sum_{j=k-N}^{N} \begin{Bmatrix} \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}_{N-j+1}^2}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{\dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)}\right\} & ; k = N+t+1 \\ \hat{c}(j,N\text{-}j\text{+}2)^2 \times \dfrac{\hat{\sigma}_{N-j+1}^2}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{\alpha_{N-j+1}^{t+1} \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)}\right\} & ; k > N+t+1 \end{Bmatrix} \quad (5.8)$$

The formulas for the total uncertainty are shown in (5.9).

$$Var\left[\widehat{CF}_t^1(tot)\right] = \sum_{k}^{N+n} \begin{Bmatrix} Var\left[\widehat{CF}^2(k)\right] + 2\hat{c}(k-N+1,n) \times \left(\sum_{i=k-N+2}^{N} c(i,n)\right) \\ \times \left[\dfrac{\hat{\sigma}_{k-N+1}^2}{\hat{F}(k-N+1)^2} \times \dfrac{1}{\sum_{j=1}^{w-1} c(j,k-N+1)}\right] & ; k = N+t+1 \\ +\left[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2\right] \times \dfrac{\hat{\sigma}_{N-j+1}^2}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{\dfrac{1}{\hat{c}(j,N\text{-}j\text{+}1)} + \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)}\right\} \\ Var\left[\widehat{CF}^2(k)\right] + 2\hat{c}(k-N+1,n) \times \left(\sum_{i=k-N+2}^{N} c(i,n)\right) \\ \times \left[\sum_{d=k-N+2}^{n-1} \left(\dfrac{\hat{\sigma}_d^2}{\hat{F}(d)^2} \times \alpha_d^{t+1} \times \dfrac{1}{\sum_{j=1}^{N-d} c(j,d)}\right)\right] & ; k > N+t+1 \\ +\left[\hat{c}(j,n)^2 - \hat{c}(j,N\text{-}j\text{+}2)^2\right] \times \dfrac{\hat{\sigma}_{N-j+1}^2}{\hat{F}(N\text{-}j\text{+}1)^2} \times \left\{\alpha_{N-j+1}^{t+1} \times \dfrac{1}{\sum_{i=1}^{N-j-1} c(i,N\text{-}j\text{+}1)}\right\} \end{Bmatrix} \quad (5.9)$$

Using formulas (5.8) and (5.9), the results for the sample data triangle will be similar to the results shown in Table 5.5, meaning the first diagonal will be equal to the same diagonal in Table 3.6 and the totals will match the same time horizon in Tables 5.1 and 5.2.

5.4 Comparison with Mack

Now that we have revised the formulas related to the Merz-Wüthrich models, it is instructive to compare the runoff for the two models using the totals from Tables 3.3, 3.4, 3.5, 5.1, and 5.2. As shown in Table 5.6, at time $t = 0$ (and $T = 1$) the standard deviation for the 1-year time horizon is 72.7% of the standard deviation for the ultimate time horizon. As previously discussed, this makes sense since the 1-year time horizon only includes the parameter variance beyond the first diagonal. In addition, the standard deviations for the last runoff period at time $t = 8$ are identical since there is no future diagonals at that point in time.

Table 5.6 – Comparison of Alternative Estimated Runoff with Mack Model

		$\hat{R}_t(tot)$	$\sqrt{Var[\hat{R}_t(tot)]}$	CoV	$\sqrt{Var[\hat{R}_t^1(tot)]}$	CoV	$Ratio$
$t =$	0	18,680,856	2,447,095	13.1%	1,778,968	9.5%	72.7%
	1	13,454,320	1,788,912	13.3%	1,258,989	9.4%	70.4%
	2	9,274,925	1,340,940	14.5%	987,439	10.6%	73.6%
	3	6,143,258	954,131	15.5%	713,534	11.6%	74.8%
	4	4,015,986	663,602	16.5%	521,112	13.0%	78.5%
	5	2,454,107	431,762	17.6%	353,057	14.4%	81.8%
	6	1,276,363	263,362	20.6%	214,796	16.8%	81.6%
	7	532,076	159,952	30.1%	144,746	27.2%	90.5%
	8	86,555	70,421	81.4%	70,421	81.4%	100.0%

In contrast to the runoff comparison with Merz-Wüthrich in Table 4.8, it does appear as though the runoff of the time horizon standard deviations adhere to the concepts used for Solvency II and, thus, is not an approximation for the runoff of the capital requirement.

5.5 Comparison of Risk Margins

As a final comparison, we can test how the different runoffs of the capital requirement affect the risk margins using the cost of capital method under Solvency II. Starting with the runoff from the Merz-Wüthrich method from Table 4.8, in Table 5.7 the lognormal distribution assumption is used to calculate the 99.5% Value at Risk (VaR). Using the VaR for each future year in the runoff, the costs of capital are calculated assuming an expected return of 6.0% and then the runoff of the cost of capital is discounted at 2.0%. Summing the discounted cost of capital over the runoff period results in a total discounted cost of capital of 891,587, which is 4.8% of the unpaid claims (i.e., 18,680,856) at $t = 0$.

Table 5.7 – Calculation of Risk Margin using Merz-Wüthrich Model

		$\hat{R}_t(tot)$	$\sqrt{Var[\hat{R}^{T'}(tot)]}$	99.5th Percentile	99.5% VaR	6.0% CoC	Discounted CoC
$t=$	0	18,680,856	1,778,968	23,753,426	5,072,570	304,354	301,328
	1	13,454,320	1,177,727	16,785,734	3,331,414	199,885	193,982
	2	9,274,925	885,178	11,799,479	2,524,553	151,473	144,092
	3	6,143,258	607,736	7,882,818	1,739,561	104,374	97,323
	4	4,015,986	428,681	5,252,966	1,236,980	74,219	67,836
	5	2,454,107	267,503	3,227,797	773,690	46,421	41,590
	6	1,276,363	128,557	1,645,023	368,659	22,120	19,425
	7	532,076	96,764	833,102	301,026	18,062	15,548
	8	86,555	49,055	293,233	206,679	12,401	10,464
Total							**891,587**
Percent of Unpaid Claims:							*4.8%*

In Table 5.8, the runoff using the alternative model from Table 5.6 is used to calculate the discounted cost of capital. Using all of the same assumptions noted above for Table 5.7, except for the standard deviation of the unpaid claims, the alternative model estimates the total discounted cost of capital at 1,007,157 or 5.4% of the unpaid claims at $t = 0$.

Table 5.8 – Calculation of Risk Margin using Alternative Model

		$\hat{R}_t(tot)$	$\sqrt{Var[\hat{R}_t^1(tot)]}$	99.5th Percentile	99.5% VaR	6.0% CoC	Discounted CoC
$t=$	0	18,680,856	1,778,968	23,753,426	5,072,570	304,354	301,328
	1	13,454,320	1,258,989	17,038,055	3,583,735	215,024	208,674
	2	9,274,925	987,439	12,123,409	2,848,484	170,909	162,580
	3	6,143,258	713,534	8,222,165	2,078,907	124,734	116,308
	4	4,015,986	521,112	5,555,442	1,539,456	92,367	84,424
	5	2,454,107	353,057	3,512,025	1,057,918	63,475	56,868
	6	1,276,363	214,796	1,935,777	659,413	39,565	34,745
	7	532,076	144,746	1,021,830	489,754	29,385	25,295
	8	86,555	70,421	421,013	334,458	20,067	16,933
Total							**1,007,157**
Percent of Unpaid Claims:							*5.4%*

Comparing Table 5.7 and 5.8, it makes sense that the risk margin is larger for the alternative method since the runoff is a bit slower.[17] While there could be situations where the alternative method results in a faster runoff and a smaller risk margin, it seems like the most common result would be for the alternative method to result in a larger risk margin. In other words, in most situations the risk margin using the Merz-Wüthrich approximation for the runoff would be underestimated.

6. CONCLUSIONS

After reviewing the Mack and Merz-Wüthrich model formulas, the paper expands their usefulness by adding runoff and cash flow formulas. By comparing the runoff of the Merz-Wüthrich results to the Mack runoff it was demonstrated the Merz-Wüthrich does reconcile with the Mack in the sense that the variances of the time windows total to the Mack variances for the ultimate time horizon and is consistent in the context of a $t = 0$ view. However, to estimate the runoff of the required capital for the cost of capital method of calculating the risk margin under Solvency II, this formula would underestimate the risk when we consider the view at $t > 0$. In order to estimate the risk for $t > 0$, the first future year for each runoff period must include both the full process and parameter variance. Thus, an alternative set of formulas were derived and demonstrated to be consistent with concepts used for Solvency II. Finally, alternate views of the covariance adjustment were developed for all of the formulas that result in a smoother transition of the coefficients of variation and aide in comparisons to other models.

[17] In this example, the risk margin is 13.0% larger but other examples could result in larger or smaller differences between the models.

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Supplementary Material

There are companion files designed to give the reader a deeper understanding of the formulas discussed in the paper and that were used to calculate all of the tables in this paper. The files are all in the "Mack & Merz-Wüthrich.zip" file. The files are:

Mack & Merz-Wüthrich Runoff.xlsm – this file contains the detailed calculations described in this paper for a single segment or line of business for a 10 x 10 triangle only. Data can be entered for a new triangle, exposures, a tail factor, and tail standard deviation.

Mack & Merz-Wüthrich Calc.xlsm – this file contains VBA functions that replicate all of the calculations in the "Runoff" file for a segment or line of business for any size triangle. Data can be entered for a new triangle, exposures, a tail factor, and tail standard deviation.

Milliman Mind Application – this file replicates all of the options in the "Runoff" Excel file using the Milliman Mind platform. The app is free to use.

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Abbreviations and notations

Collect here in alphabetical order all abbreviations and notations used in the paper

CL, chain ladder
CoV, coefficient of variation
CVA, covariance adjustment
MSEP, mean squared error of prediction
VaR, Value at Risk

Biography of the Author

Mark R. Shapland is a Principal & Consulting Actuary in Milliman's Dubai office where he is responsible for various reserving and pricing projects for a variety of clients and was previously the lead actuary for the Property & Casualty Insurance Software (PCIS) development team. He has a B.S. degree in Integrated Studies (Actuarial Science) from the University of Nebraska-Lincoln. He is a Fellow of the Casualty Actuarial Society, a Fellow of the Society of Actuaries and a Member of the American Academy of Actuaries. He was the leader of Section 3 of the Reserve Variability Working Party, the Chair of the CAS Committee on Reserves, co-chair of the Tail Factor Working Party, and co-chair of the Loss Simulation Model Working Party. He is also a co-developer and co-presenter of the CAS Reserve Variability Limited Attendance Seminar, the European Actuarial Academy's Stochastic Modelling Seminar, and has spoken frequently on this subject both within the CAS and internationally. He can be contacted at mark.shapland@milliman.com.

